

$U(2)_A \times U(2)_V$ -symmetric fixed point from the Functional Renormalization Group

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The existence of an $U(2)_A \times U(2)_V$ -symmetric fixed point in the chiral linear sigma model is confirmed using the Functional Renormalization Group (FRG). Its stability properties and the implications for the order of the chiral phase transition of two-flavor quantum chromodynamics (QCD) are discussed. Furthermore, several technical conclusions are drawn from the comparison with the results of resummed loop expansions.

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I. INTRODUCTION

Apart from other methods, our current understanding of QCD in the nonperturbative regime is strongly based on lattice gauge theory and effective models [1]. These complementary approaches are compared to each other for reasons of crosscheck and systematic improvement [2, 3]. Despite all efforts the order of the chiral phase transition of QCD with two massless flavors has not been rigorously determined yet, and the interest in a reliable prediction remains strong. The case of two massless (or light, respectively) flavors at vanishing baryonic chemical potential is of particular interest for lattice studies due to the comprehensive predictions of effective models [4, 5]. The possible existence of a second-order chiral phase transition, as well as the corresponding universality class, can be investigated from the effective theory for the chiral condensate [6–18]. Using the $[\bar{2}, 2] + [2, \bar{2}]$ representation of $SU(2)_L \times SU(2)_R$ [19], we can take into account the scalar mesons (σ and \vec{a}_0) as well as the pseudoscalar mesons (η and $\vec{\pi}$) by writing down the most general Lagrangian invariant under chiral symmetry. For the full symmetry, $U(2)_A \times U(2)_V \simeq U(1)_A \times U(1)_V \times [SU(2)/Z(2)]_L \times [SU(2)/Z(2)]_R$, taking into account all linearly independent invariants up to eighth polynomial order in the fields, this Lagrangian can be written as [6, 7, 20, 21]

$$\begin{aligned} \mathcal{L} = & \frac{Z}{2} \text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r \text{Tr} \Phi^\dagger \Phi + g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + g_2 \xi + g_3 (\text{Tr} \Phi^\dagger \Phi)^3 \\ & + g_4 (\text{Tr} \Phi^\dagger \Phi) \xi + g_5 \text{Tr}(\Phi^\dagger \Phi)^4 + g_6 (\text{Tr} \Phi^\dagger \Phi)^4 + g_7 (\text{Tr} \Phi^\dagger \Phi)^2 \text{Tr}(\Phi^\dagger \Phi)^2, \end{aligned} \quad (1)$$

where $\Phi = (\sigma + i\eta) t_0 + \vec{t} \cdot (\vec{a}_0 + i\vec{\pi})$, with t_a denoting the generators of $U(2)$ normalized such that $\text{Tr}(t_a t_b) \equiv 1$ [13]. Furthermore,

$$\begin{aligned} \text{Tr} \Phi^\dagger \Phi &= \sum_i \phi_i^2 \equiv 2\rho, \quad \phi_i \equiv \sigma, \vec{\pi}, \eta, \vec{a}_0, \\ \frac{1}{2} \text{Tr}(\Phi^\dagger \Phi)^2 - \rho^2 &= (\sigma^2 + \vec{\pi}^2)(\eta^2 + \vec{a}_0^2) - (\sigma\eta - \vec{\pi} \cdot \vec{a}_0)^2 \equiv \xi. \end{aligned}$$

We omit derivative couplings since we will only discuss the local-potential approximation (LPA, $Z = 1$) and, respectively, its minimal extension allowing for a field-independent wave-function renormalization factor Z (LPA'). We note that the invariants $(\text{Tr} \Phi^\dagger \Phi) \text{Tr}(\Phi^\dagger \Phi)^3$, $(\text{Tr}(\Phi^\dagger \Phi)^2)^2$, and $\text{Tr}(\Phi^\dagger \Phi)^3$ do not yield further linearly independent contributions to Eq. (1).

In this paper we focus on the case where the axial $U(1)_A$ symmetry, which is anomalously broken at vanishing temperature, has already been restored at the critical temperature T_c . Therefore we do not take account of $U(1)_A$ -breaking terms in Eq. (1). For studies concerning the opposite scenario in which the anomaly remains present at T_c we refer to Refs. [13–16, 22–25]. The long-standing question which of the both scenarios is actually realized is subject to an ongoing debate. The latest lattice results are quite controversial: whereas the case of restored anomaly is advocated by Refs. [4, 26], the opposite scenario is favored by Refs. [5, 27]. The predictions of effective theories for the chiral condensate are summarized in the following.

The existence of an infrared-stable (IR-stable) fixed point in the RG flow of the effective theory for the order parameter is a necessary condition for a second-order phase transition to occur. If this scenario is realized or not depends on the initial values for the parameters in the ultraviolet (UV) limit determined by the underlying microscopic theory. Therefore, the RG analysis serves to either rule out the existence of a second-order phase transition or to confirm its possible existence.

If the anomaly strength exceeds the cut-off scale, a phase transition of second order in the $O(4)$ universality class is predicted [6, 13, 28]. The case of small anomaly strength is subtle. The anomaly yields two independent quadratic mass terms. In Landau theory, i.e., at mean-field level, it is evident that such a situation corresponds to a multicritical point with at least two relevant scaling variables. This is used as an argument in Ref. [14] to rule out a second-order phase transition with temperature being the only relevant scaling variable. However, in consistence with Refs. [13, 20], we argue that the inclusion of fluctuations can, in principle, lead to a IR-stable fixed point corresponding to exactly such a scenario. Although associated with unphysical masses in the approximation considered, there in fact exists an (unphysical) $SU(2)_A \times U(2)_V$ -symmetric, IR-stable fixed point exemplifying our consideration. This observation extends the critical reinvestigation of the standard criterions used for ruling out continuous transitions presented in Ref. [29]. The latter particularly points out that the irreducibility of a representation is not strictly ruling out a second-order phase transition associated with a single relevant scaling variable. In the absence of the anomaly there is strong evidence from Refs. [14, 17] for the existence of a second-order phase transition belonging to the $U(2)_V \times U(2)_A$ universality class. The existence and properties of the corresponding fixed point will be discussed in the remainder of this paper.

Ref. [14] uses a resummed loop expansion at fixed spatial dimension, $D = 3$, based on the \overline{MS} and the MZM scheme, respectively. The discovered IR-stable, $U(2)_V \times U(2)_A$ -symmetric fixed point corresponds to an anomalous dimension of $\eta \sim 0.12$. Previous studies in the framework of the ϵ -expansion ($\epsilon = 4 - D$) failed to find the fixed point [6, 10]. It is an important question why this is the case. A plausible explanation is given in Ref. [14]: the fixed point only exists near $D = 3$. One might wonder, however, if the resummation scheme and the loop-order also play a role. With our FRG investigation presented in Secs. II–III we demonstrate that the existence not only depends on the fixed spatial dimension, but also on the way how nonperturbative corrections are included.

Due to the converging correlation length at a second-order phase transition we can work in the dimensionally reduced theory [30].

II. FIXED POINTS FROM FRG

Assuming an homogeneous condensate, and using the Litim regulator, the Wetterich equation for the potential of the truncation (1) is given by

$$\frac{\partial U_k}{\partial k} = \frac{2\pi^{D/2}k^{D+1}Z_k}{D\Gamma(D/2)(2\pi)^D} \left(1 - \frac{\eta}{2+D}\right) \sum_i \frac{1}{Z_k k^2 + M_i^2}, \quad (2)$$

where $\mathcal{L}_k = \frac{1}{2}Z_k \text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + U_k$, with $\mathcal{L}_{k=\Lambda} = \mathcal{L}$ defining the bare Lagrangian in the UV limit. M_i^2 denote the eigenvalues of the mass matrix

$$M_{ij} \equiv \frac{\partial^2 U_k}{\partial \phi_i \partial \phi_j}, \quad i, j = 1, \dots, 8. \quad (3)$$

The anomalous dimension, η , is determined from the relation

$$\eta_k = -Z_k^{-1}k \frac{\partial Z_k}{\partial k}, \quad \lim_{k \rightarrow 0} \eta_k = \eta. \quad (4)$$

The flow equation for Z_k is derived from the second derivative of the effective action with respect to the fields and evaluated at the global minimum of the potential [31]. For our purposes we can restrict our discussion of the LPA' to the truncation $U_k(\rho, \xi) \equiv V(\rho) + W(\rho)\xi$, which is suited up to sextic truncation order ($g_5 = g_6 = g_7 = 0$). Setting $D = 3$, in agreement with Ref. [32] we obtain

$$\eta_k = \frac{2}{3\pi^2[1 + \bar{V}'_k(\bar{\rho}_{0,k})]^2} \left(\frac{4\bar{\rho}_{0,k}\bar{W}_k(\bar{\rho}_{0,k})^2}{[1 + 4\bar{W}_k(\bar{\rho}_{0,k})\bar{\rho}_{0,k} + \bar{V}'_k(\bar{\rho}_{0,k})]^2} + \frac{\bar{\rho}_{0,k}\bar{V}''_k(\bar{\rho}_{0,k})^2}{[1 + \bar{V}'_k(\bar{\rho}_{0,k}) + 2\bar{\rho}_{0,k}\bar{V}''_k(\bar{\rho}_{0,k})]^2} \right), \quad (5)$$

where we introduced rescaled variables (labeled by a bar),

$$\bar{U} = k^{-D}U, \quad \bar{\rho} = Zk^{2-D}\rho, \quad \bar{\xi} = Z^2k^{4-2D}\xi, \quad \bar{V} = k^{-D}V, \quad \bar{W} = Z^{-2}k^{D-4}W,$$

and denoted the global minimum of U_k by ρ_0 (assuming $\xi_0 = 0$).

The flow equations for the rescaled parameters of Eq. (1) are derived similar to Refs. [12, 13], not listed explicitly here. The numerically determined fixed points for sextic truncation order are listed in Table I, those for octic truncation order in Table II. We proceed with a detailed analysis of their stability properties and the resultant implications in Sec. III.

III. STABILITY ANALYSIS

In order to determine the stability properties of the fixed points one can analyze the flow in their neighborhood where it is governed by the linearized system. For this purpose one calculates the eigenvalues of the stability matrix

$$(S_{ij}) \equiv \left(\frac{\partial \beta_i}{\partial \bar{p}_j} \right) \Big|_{\bar{p}=\bar{p}^*}, \quad (6)$$

where we denote the n rescaled parameters of the Lagrangian by $\bar{p} = \{\bar{p}_i\}$, a fixed point by $\{\bar{p}_i^*\}$, and the beta functions are given by $\beta_i(\bar{p}) \equiv k\partial_k \bar{p}_i$. In general one obtains n_s eigenvalues with

TABLE I: Fixed points in sextic truncation order (for the LPA denoted by $F_i^{(6)}$, for the LPA' by $F_i'^{(6)}$). $D = 3$.

F	\bar{r}	\bar{g}_1	\bar{g}_2	\bar{g}_3	\bar{g}_4	η
$F_0^{(6)}, F_0'^{(6)}$	0	0	0	0	0	0
$F_1^{(6)}$	-0.1316	0.0827	0.8586	0.2091	0.2161	0
$F_1'^{(6)}$	-0.1251	0.0795	0.8447	0.1981	0.1876	0.0334
$F_2^{(6)}$	-0.103	0.3334	-0.9411	0.307	-0.7154	0
$F_2'^{(6)}$	-0.0938	0.3151	-0.8981	0.2634	-0.6024	0.0529
$F_3^{(6)}$	-0.1355	0.2132	0	0.1285	0	0
$F_3'^{(6)}$	-0.1317	0.2103	0	0.123	0	0.0195

F	stability-matrix eigenvalues	nonzero \tilde{M}_i^2
$F_0^{(6)}, F_0'^{(6)}$	$\{-2, -1, -1, 0, 0\}$	—
$F_1^{(6)}$	$\{15.6603, 0.6245+3.5342i, 0.6245-3.5342i, 1.6306, -1.3743\}$	$\{0.8246, 0.6434, 0.6434, 0.6434\}$
$F_1'^{(6)}$	$\{14.5059, 0.5839+3.2722i, 0.5839-3.2722i, 1.5485, -1.3614\}$	$\{0.7823, 0.6261, 0.6261, 0.6261\}$
$F_2^{(6)}$	$\{13.2219, 1.1882+2.1481i, 1.1882-2.1481i, -1.5108, 1.3732\}$	$\{0.4750, -0.2707, -0.2707, -0.2707\}$
$F_2'^{(6)}$	$\{11.6716, 1.0622+1.874i, 1.0622-1.874i, -1.5279, 1.3464\}$	$\{0.4272, -0.2502, -0.2502, -0.2502\}$
$F_3^{(6)}$	$\{12.9247, 8.125, 1.5092, -1.3798, -0.5034\}$	$\{0.6442\}$
$F_3'^{(6)}$	$\{12.3598, 7.7931, 1.4673, -1.3745, -0.4802\}$	$\{0.6233\}$

TABLE II: Fixed points for the LPA in octic truncation order. $D = 3$.

F	\bar{r}	\bar{g}_1	\bar{g}_2	\bar{g}_3	\bar{g}_4	\bar{g}_5	\bar{g}_6	\bar{g}_7
$F_0^{(8)}$	0	0	0	0	0	0	0	0
$F_1^{(8)}$	-0.0153	0.0274	0.1007	-0.0020	-0.1529	-0.0432	-0.0143	0.0321
$F_2^{(8)}$	-0.0141	0.0567	-0.1151	-0.0485	0.1676	-0.0472	-0.0997	0.1429
$F_3^{(8)}$	-0.0148	0.0414	0	-0.0258	0	0	-0.0118	0
$F_4^{(8)}$	-0.1721	0.2192	0	0.1828	0	0	0.1006	0

F	stability-matrix eigenvalues
$F_0^{(8)}$	$\{-2, -1, -1, 1, 1, 1, 0, 0\}$
$F_1^{(8)}$	$\{4.1374, 2.1731, -2.0064, 1.3755, -1.1678, -0.9261, 0.2814, 0.1298\}$
$F_2^{(8)}$	$\{4.2123, 2.3531, -2.0072, 1.5805, -1.1886, -0.9334, 0.2274, 0.1216\}$
$F_3^{(8)}$	$\{3.6611, 2.8933, -2.0029, 1.685, -1.1024, -1.0081, 0.1351, -0.0505\}$
$F_4^{(8)}$	$\{34.5986, 26.9475, 12.7525, 9.3877, 5.1825, 1.3058, -1.1215, -0.65\}$

positive real part, n_u with negative real part, and n_m with vanishing real part. The corresponding eigenvectors give rise to invariant subspaces of the parameter space inside which the flow stays if one starts within them [33]. In case of distinct eigenvalues there is a n_s -dimensional invariant subspace (called critical manifold) inside which the flow is attracted towards the fixed point in the infrared limit $k = 0$. Respectively, there exists a n_u -dimensional invariant subspace (called unstable manifold) inside which the flow is repelled, and a n_m -dimensional invariant subspace (called marginal manifold) inside which the flow has no direction at all. Here we note that complex valued eigenvalues always appear as conjugate pairs. Referring to the real and imaginary

parts of the associated complex eigenvectors as eigenvectors, too, the critical manifold is spanned by n_s eigenvectors, the unstable manifold is spanned by n_u eigenvectors, and the marginal manifold is spanned by n_m eigenvectors. Therefore, if $n_m = 0$, one can reach the critical manifold by tuning n_u parameters starting anywhere in parameter space. Hence, a second-order phase transition with respect to a single scaling variable (temperature) can only exist if we have exactly $n_u = 1$. In this case we speak of an IR-stable fixed point.

The stability-matrix eigenvalues are listed for each fixed point in Table I–II. We begin with discussing the LPA in sextic truncation order. $F_1^{(6)}$ and $F_2^{(6)}$ are different, IR-stable, $U(2)_A \times U(2)_V$ -symmetric spiral fixed points. $F_1^{(6)}$ is associated with physical mass-matrix eigenvalues whereas $F_2^{(6)}$ is not. Their existence is highly nontrivial since they do not exist at quartic truncation order, neither in the LPA [12, 18], nor in the LPA' [32]. The critical exponent, $\nu \sim 1/1.3614 \sim 0.7345$, associated with $F_1'^{(6)}$ is in unexpectedly good agreement with the values reported in Ref. [14] ($\nu \sim 0.71$ for the MZM scheme, $\nu \sim 0.76$ for the \overline{MS} scheme). This agreement is most likely accidental. The value for the anomalous dimension is actually significantly smaller ($\eta \sim 0.0334$ compared to $\eta \sim 0.1$). $F_3^{(6)}$ is an unstable $O(8)$ -symmetric fixed point. All fixed points are also present in the LPA' without qualitative changes ($F_i^{(6)}$ corresponds to $F_i'^{(6)}$).

Of particular interest to us are the marginal eigenvalues encountered for the Gaussian fixed points, F_0 , which will be discussed next. From a merely mathematical standpoint one can decide whether marginal eigenvalues are relevant or not by going beyond the linear order utilized in Eq. (6). For this purpose one can either use the second derivatives of the beta functions, or one has to perform a more general Lyapunov analysis. However, this is not meaningful in our case because in presence of marginal eigenvalues one has to consider a change in the fixed-point structure at higher polynomial truncation order. In general, such a change cannot be excluded by a nonlinear stability analysis at lower order. The occurrence of the marginal eigenvalues, however, can be explained as follows. In general the beta functions for a rescaled mass parameter \bar{m}^2 , a rescaled quartic coupling \bar{g}_4 , and a rescaled sextic coupling \bar{g}_6 , respectively, are given by

$$\beta_{m^2} = (-2 + \eta)\bar{m}^2 + f_2(\bar{p}) , \quad \beta_4 = (D - 4 + 2\eta)\bar{g}_4 + f_4(\bar{p}) , \quad \beta_6 = (2D - 6 + 3\eta)\bar{g}_6 + f_6(\bar{p}) , \quad (7)$$

where the $f_i(\bar{p})$ denote nonlinear functions of the rescaled parameters. In FRG the polynomial order of these functions depends on the truncation order of the effective action, whereas in RG approaches based on a loop expansion it depends on the loop order. Since these functions as well as the anomalous dimension, η , vanish at the Gaussian fixed point, we can conclude that (for $D = 3$) \bar{m}^2 and \bar{g}_4 are relevant parameters with respect to this fixed point. They yield stability matrix eigenvalues -2 and -1 , respectively. Similarly, the sextic coupling contributes a vanishing eigenvalue at the Gaussian fixed point, and higher order couplings yield positive eigenvalues. We conclude that the marginal eigenvalues in Table I–II do not render the stability analysis inconclusive. However, in the remainder of this section, we will argue why the LPA' remains inconclusive, pointing out general differences between FRG and other RG approaches first. For a more fundamental comparison between both approaches we refer to Refs. [34, 35].

In the framework of the ϵ -expansion or other loop expansions at fixed spatial dimension D , one usually argues that also in case of non-Gaussian fixed points the canonical scaling dimension determines if a coupling can affect stability [36]. Accordingly, depending on the sign of their canonical scaling dimension, one speaks of relevant, marginal, and irrelevant parameters. Obviously, especially marginal eigenvalues are sensitive to the loop order. Therefore, one has to consider the possibility that higher-order loop corrections change the marginal eigenvalue into a

nonvanishing one. It is important to note that if a marginal eigenvalue for a certain fixed point turns nonzero at higher order, this can also change the stability properties of the other fixed points. This is for example the case in the $O(N=4)$ model with di-icosahedral anisotropy. The ϵ -expansion of this model has been derived in Ref. [37], pointing out that the case of $N=4$ is special. In the presence of an anisotropy, the $O(4)$ -symmetric fixed point acquires a marginal eigenvalue at one-loop order in the ϵ -expansion whereas the anisotropic fixed point is IR-unstable. At two-loop order, however, the anisotropic fixed point can become the IR-stable one. We reinvestigated the situation using the FRG in LPA and found that the anisotropic fixed point also becomes IR-stable when going beyond the quartic truncation order [20].

However, a change of stability can occur even in the absence of any marginal eigenvalues. A famous example is the $O(N)$ model with cubic anisotropy for $D=3$ [38, 39]. The model exhibits an $O(N)$ -symmetric (isotropic) fixed point as well as a cubic fixed point. For $N > N_c$ the cubic fixed point is the IR-stable one, the isotropic fixed point being IR-unstable, and vice versa for $N < N_c$. The value for N_c depends on the loop order as well as on the resummation scheme and is still under debate.

In comparison to loop expansions, the stability matrix eigenvalues are much more sensitive to the polynomial truncation order in the FRG formalism. Using FRG, the accuracy of the critical exponents heavily depends on irrelevant couplings [40]. This is explained by the fact that fluctuations are taken into account differently in both approaches. Irrelevant couplings can be safely ignored in the loop expansion and nonperturbative effects are captured by using resummation. In contrast, if we were able to solve the FRG equation without truncating the effective action, we would obtain exact results. In the LPA at quartic truncation order, however, one generically reproduces the one-loop epsilon-expansion results when setting the mass parameter to zero [12, 31].

Our conclusions are as follows. Naively, one would trust the utilized approximation scheme since no marginal eigenvalues appear for the non-Gaussian fixed points. However, we argued that even in this case the fixed-point structure can change at higher truncation order. Especially the presence of the unphysical fixed point advises caution. In fact, the spiral fixed points become unstable fixed points ($F_1^{(8)}$ and $F_2^{(8)}$, respectively) at octic truncation order (see Table II). Interestingly, at this order one finds two unstable $O(8)$ -symmetric fixed points. Going to any higher (finite) polynomial order in the LPA' will not clarify the situation. If an IR-stable fixed point were found at higher order, one could not rule out its disappearance beyond that order. And in the opposite case the discrepancy with Ref. [14] would require to go beyond the LPA' as well. Therefore it is necessary to include derivative couplings in order to decide whether the $U(2)_A \times U(2)_V$ -symmetric fixed point is stable or not. In addition, novel criteria to assess the conclusiveness of truncation schemes need to be developed.

IV. CONCLUSIONS

We further investigated the possibility that the two-flavor chiral phase transition can be of second order in the absence of the axial anomaly, using the FRG method in the LPA as well as in the LPA'.

We found two IR-stable, $U(2)_A \times U(2)_V$ -symmetric fixed points at sextic polynomial truncation order, one of them associated with unphysical masses. The value for the critical exponent,

$\nu \sim 0.7345$, calculated for the one associated with physical masses is in (most likely accidental) agreement with the result reported in Ref. [14]. At higher polynomial order both fixed points become unstable. Nevertheless, the results of our research provide further evidence for the existence of the IR-stable, $U(2)_A \times U(2)_V$ -symmetric fixed point from an independent perspective. The fact that an $U(2)_A \times U(2)_V$ -symmetric fixed point appears by simply including sextic invariants demonstrates that its existence not only depends on the spatial dimension but also on the way nonperturbative corrections are taken into account. In the framework of a resummed perturbative expansion this concerns the resummation scheme and the perturbative order. Our main conclusion is that the LPA' is not capable to unambiguously clarify the stability of the fixed points. Since the fixed-point structure of the dimensionally reduced theory controls the behavior near T_c , previous finite-temperature studies [18, 20, 32] remain inconclusive, too. We expect clarification beyond the LPA' taking into account derivative couplings. Finally, the simultaneous occurrence of two IR-stable fixed points (although one of them being unphysical, and the truncation is not reliable) is interesting regarding the universality hypothesis. The example illustrates that, in principle, it is possible that two systems sharing (a) the same spatial dimension, (b) the same number of order parameter components, and (c) the same symmetry properties can be attracted to different IR-stable fixed points (here $F_1'^{(6)}$ and $F_2'^{(6)}$, respectively). Both associated universality classes are characterized by the same representation of the same symmetry group. However, we state clearly that the given example has to be regarded as an artifact of the utilized truncation. A similar situation, although to our knowledge not strictly ruled out, is commonly not believed to appear in a physical setting.

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